

LEARNING OBJECTIVES

In this section, you will:

- Convert from logarithmic to exponential form.
- Convert from exponential to logarithmic form.
- Evaluate logarithms.
- Use common logarithms.
- Use natural logarithms.
- Identify the domain of a logarithmic function.
- Graph logarithmic functions.



Figure 1 Devastation of March 11, 2011 earthquake in Honshu, Japan. (credit: Daniel Pierce)

In 2010, a major earthquake struck Haiti, destroying or damaging over 285,000 homes^[19]. One year later, another, stronger earthquake devastated Honshu, Japan, destroying or damaging over 332,000 buildings,^[20] like those shown in **Figure 1**. Even though both caused substantial damage, the earthquake in 2011 was 100 times stronger than the earthquake in Haiti. How do we know? The magnitudes of earthquakes are measured on a scale known as the Richter Scale. The Haitian earthquake registered a 7.0 on the Richter Scale^[21] whereas the Japanese earthquake registered a 9.0.^[22]

The Richter Scale is a base-ten logarithmic scale. In other words, an earthquake of magnitude 8 is not twice as great as an earthquake of magnitude 4. It is $10^{8-4} = 10^4 = 10,000$ times as great! In this lesson, we will investigate the nature of the Richter Scale and the base-ten function upon which it depends.

Converting from Logarithmic to Exponential Form

In order to analyze the magnitude of earthquakes or compare the magnitudes of two different earthquakes, we need to be able to convert between logarithmic and exponential form. For example, suppose the amount of energy released from one earthquake were 500 times greater than the amount of energy released from another. We want to calculate the difference in magnitude. The equation that represents this problem is $10^x = 500$, where x represents the difference in magnitudes on the Richter Scale. How would we solve for x ?

We have not yet learned a method for solving exponential equations. None of the algebraic tools discussed so far is sufficient to solve $10^x = 500$. We know that $10^2 = 100$ and $10^3 = 1000$, so it is clear that x must be some value between 2 and 3, since $y = 10^x$ is increasing. We can examine a graph, as in **Figure 2**, to better estimate the solution.

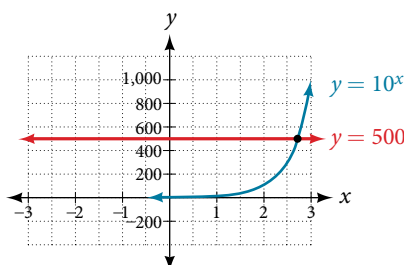


Figure 2

19 <http://earthquake.usgs.gov/earthquakes/eqinthenews/2010/us2010rja6/#summary>. Accessed 3/4/2013.

20 <http://earthquake.usgs.gov/earthquakes/eqinthenews/2011/usc001xgp/#summary>. Accessed 3/4/2013.

21 <http://earthquake.usgs.gov/earthquakes/eqinthenews/2010/us2010rja6/>. Accessed 3/4/2013.

22 <http://earthquake.usgs.gov/earthquakes/eqinthenews/2011/usc001xgp/#details>. Accessed 3/4/2013.

Estimating from a graph, however, is imprecise. To find an algebraic solution, we must introduce a new function. Observe that the graph in **Figure 2** passes the horizontal line test. The exponential function $y = b^x$ is one-to-one, so its inverse, $x = b^y$ is also a function. As is the case with all inverse functions, we simply interchange x and y and solve for y to find the inverse function. To represent y as a function of x , we use a logarithmic function of the form $y = \log_b(x)$. The base b **logarithm** of a number is the exponent by which we must raise b to get that number.

We read a logarithmic expression as, “The logarithm with base b of x is equal to y ,” or, simplified, “log base b of x is y .” We can also say, “ b raised to the power of y is x ,” because logs are exponents. For example, the base 2 logarithm of 32 is 5, because 5 is the exponent we must apply to 2 to get 32. Since $2^5 = 32$, we can write $\log_2 32 = 5$. We read this as “log base 2 of 32 is 5.”

We can express the relationship between logarithmic form and its corresponding exponential form as follows:

$$\log_b(x) = y \Leftrightarrow b^y = x, b > 0, b \neq 1$$

Note that the base b is always positive.

$$\log_b(x) = y \quad \begin{array}{l} \text{Think} \\ b \text{ to the } y = x \end{array}$$

to

Because logarithm is a function, it is most correctly written as $\log_b(x)$, using parentheses to denote function evaluation, just as we would with $f(x)$. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written without parentheses, as $\log_b x$. Note that many calculators require parentheses around the x .

We can illustrate the notation of logarithms as follows:

$$\log_b(c) = a \text{ means } b^a = c$$

to

Notice that, comparing the logarithm function and the exponential function, the input and the output are switched. This means $y = \log_b(x)$ and $y = b^x$ are inverse functions.

definition of the logarithmic function

A **logarithm** base b of a positive number x satisfies the following definition.

For $x > 0, b > 0, b \neq 1$,

$$y = \log_b(x) \text{ is equivalent to } b^y = x$$

where,

- we read $\log_b(x)$ as, “the logarithm with base b of x ” or the “log base b of x .”
- the logarithm y is the exponent to which b must be raised to get x .

Also, since the logarithmic and exponential functions switch the x and y values, the domain and range of the exponential function are interchanged for the logarithmic function. Therefore,

- the domain of the logarithm function with base b is $(0, \infty)$.
- the range of the logarithm function with base b is $(-\infty, \infty)$.

Q & A...

Can we take the logarithm of a negative number?

No. Because the base of an exponential function is always positive, no power of that base can ever be negative. We can never take the logarithm of a negative number. Also, we cannot take the logarithm of zero. Calculators may output a log of a negative number when in complex mode, but the log of a negative number is not a real number.

How To...

Given an equation in logarithmic form $\log_b(x) = y$, convert it to exponential form.

1. Examine the equation $y = \log_b(x)$ and identify b , y , and x .
2. Rewrite $\log_b(x) = y$ as $b^y = x$.

Example 1 Converting from Logarithmic Form to Exponential Form

Write the following logarithmic equations in exponential form.

a. $\log_6(\sqrt{6}) = \frac{1}{2}$ b. $\log_3(9) = 2$

Solution First, identify the values of b , y , and x . Then, write the equation in the form $b^y = x$.

a. $\log_6(\sqrt{6}) = \frac{1}{2}$ Here, $b = 6$, $y = \frac{1}{2}$, and $x = \sqrt{6}$. Therefore, the equation $\log_6(\sqrt{6}) = \frac{1}{2}$ is equivalent to $6^{\frac{1}{2}} = \sqrt{6}$.

b. $\log_3(9) = 2$ Here, $b = 3$, $y = 2$, and $x = 9$. Therefore, the equation $\log_3(9) = 2$ is equivalent to $3^2 = 9$.

Converting from Exponential to Logarithmic Form

To convert from exponents to logarithms, we follow the same steps in reverse. We identify the base b , exponent x , and output y . Then we write $x = \log_b(y)$.

Example 2 Converting from Exponential Form to Logarithmic Form

Write the following exponential equations in logarithmic form.

a. $2^3 = 8$ b. $5^2 = 25$ c. $10^{-4} = \frac{1}{10,000}$

Solution First, identify the values of b , y , and x . Then, write the equation in the form $x = \log_b(y)$.

a. $2^3 = 8$ Here, $b = 2$, $x = 3$, and $y = 8$. Therefore, the equation $2^3 = 8$ is equivalent to $\log_2(8) = 3$.

b. $5^2 = 25$ Here, $b = 5$, $x = 2$, and $y = 25$. Therefore, the equation $5^2 = 25$ is equivalent to $\log_5(25) = 2$.

c. $10^{-4} = \frac{1}{10,000}$ Here, $b = 10$, $x = -4$, and $y = \frac{1}{10,000}$. Therefore, the equation $10^{-4} = \frac{1}{10,000}$ is equivalent to $\log_{10}\left(\frac{1}{10,000}\right) = -4$.

Evaluating Logarithms

Knowing the squares, cubes, and roots of numbers allows us to evaluate many logarithms mentally. For example, consider $\log_2(8)$. We ask, “To what exponent must 2 be raised in order to get 8?” Because we already know $2^3 = 8$, it follows that $\log_2(8) = 3$.

Now consider solving $\log_7(49)$ and $\log_3(27)$ mentally.

- We ask, “To what exponent must 7 be raised in order to get 49?” We know $7^2 = 49$. Therefore, $\log_7(49) = 2$
- We ask, “To what exponent must 3 be raised in order to get 27?” We know $3^3 = 27$. Therefore, $\log_3(27) = 3$

Even some seemingly more complicated logarithms can be evaluated without a calculator. For example, let’s evaluate $\log_3\left(\frac{4}{9}\right)$ mentally.

- We ask, “To what exponent must $\frac{2}{3}$ be raised in order to get $\frac{4}{9}$?” We know $2^2 = 4$ and $3^2 = 9$, so $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$. Therefore, $\log_3\left(\frac{4}{9}\right) = 2$.

How To...

Given a logarithm of the form $y = \log_b(x)$, evaluate it mentally.

1. Rewrite the argument x as a power of b : $b^y = x$.
2. Use previous knowledge of powers of b identify y by asking, “To what exponent should b be raised in order to get x ?”

Example 3 Solving Logarithms Mentally

Solve $y = \log_4(64)$ without using a calculator.

Solution First we rewrite the logarithm in exponential form: $4^y = 64$. Next, we ask, “To what exponent must 4 be raised in order to get 64?”

We know $4^3 = 64$ therefore, $\log_4(64) = 3$.

Example 4 Evaluating the Logarithm of a Reciprocal

Evaluate $y = \log_3\left(\frac{1}{27}\right)$ without using a calculator.

Solution First we rewrite the logarithm in exponential form: $3^y = \frac{1}{27}$. Next, we ask, “To what exponent must 3 be raised in order to get $\frac{1}{27}$?”

We know $3^3 = 27$, but what must we do to get the reciprocal, $\frac{1}{27}$? Recall from working with exponents that $b^{-a} = \frac{1}{b^a}$. We use this information to write

$$\begin{aligned} 3^{-3} &= \frac{1}{3^3} \\ &= \frac{1}{27} \end{aligned}$$

Therefore, $\log_3\left(\frac{1}{27}\right) = -3$.

Using Common Logarithms

Sometimes we may see a logarithm written without a base. In this case, we assume that the base is 10. In other words, the expression $\log(x)$ means $\log_{10}(x)$. We call a base-10 logarithm a **common logarithm**. Common logarithms are used to measure the Richter Scale mentioned at the beginning of the section. Scales for measuring the brightness of stars and the pH of acids and bases also use common logarithms.

definition of the common logarithm

A **common logarithm** is a logarithm with base 10. We write $\log_{10}(x)$ simply as $\log(x)$. The common logarithm of a positive number x satisfies the following definition.

For $x > 0$,

$$y = \log(x) \text{ is equivalent to } 10^y = x$$

We read $\log(x)$ as, “the logarithm with base 10 of x ” or “log base 10 of x .”

The logarithm y is the exponent to which 10 must be raised to get x .

How To...

Given a common logarithm of the form $y = \log(x)$, evaluate it mentally.

1. Rewrite the argument x as a power of 10: $10^y = x$.
2. Use previous knowledge of powers of 10 to identify y by asking, “To what exponent must 10 be raised in order to get x ?”

Example 5 Finding the Value of a Common Logarithm Mentally

Evaluate $y = \log(1,000)$ without using a calculator.

Solution First we rewrite the logarithm in exponential form: $10^y = 1,000$. Next, we ask, “To what exponent must 10 be raised in order to get 1,000?” We know $10^3 = 1,000$ therefore, $\log(1,000) = 3$.

How To...

Given a common logarithm with the form $y = \log(x)$, evaluate it using a calculator.

1. Press **[LOG]**.
2. Enter the value given for x , followed by **[)]**.
3. Press **[ENTER]**.

Example 6 Finding the Value of a Common Logarithm Using a Calculator

Evaluate $y = \log(321)$ to four decimal places using a calculator.

Solution

- Press **[LOG]**.
- Enter 321, followed by **[)]**.
- Press **[ENTER]**.

Rounding to four decimal places, $\log(321) \approx 2.5065$.

Analysis Note that $10^2 = 100$ and that $10^3 = 1000$. Since 321 is between 100 and 1000, we know that $\log(321)$ must be between $\log(100)$ and $\log(1000)$. This gives us the following:

$$\begin{array}{ccccc} 100 & < & 321 & < & 1000 \\ 2 & < & 2.5065 & < & 3 \end{array}$$

Example 7 Rewriting and Solving a Real-World Exponential Model

The amount of energy released from one earthquake was 500 times greater than the amount of energy released from another. The equation $10^x = 500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Solution We begin by rewriting the exponential equation in logarithmic form.

$$\begin{array}{lcl} 10^x = 500 \\ \log(500) = x & \text{Use the definition of the common log.} \end{array}$$

Next we evaluate the logarithm using a calculator:

- Press **[LOG]**.
- Enter 500, followed by **[)]**.
- Press **[ENTER]**.
- To the nearest thousandth, $\log(500) \approx 2.699$.

The difference in magnitudes was about 2.699.

Using Natural Logarithms

The most frequently used base for logarithms is e . Base e logarithms are important in calculus and some scientific applications; they are called **natural logarithms**. The base e logarithm, $\log_e(x)$, has its own notation, $\ln(x)$.

Most values of $\ln(x)$ can be found only using a calculator. The major exception is that, because the logarithm of 1 is always 0 in any base, $\ln(1) = 0$. For other natural logarithms, we can use the \ln key that can be found on most scientific calculators. We can also find the natural logarithm of any power of e using the inverse property of logarithms.

definition of the natural logarithm

A **natural logarithm** is a logarithm with base e . We write $\log_e(x)$ simply as $\ln(x)$. The natural logarithm of a positive number x satisfies the following definition.

For $x > 0$,

$$y = \ln(x) \text{ is equivalent to } e^y = x$$

We read $\ln(x)$ as, “the logarithm with base e of x ” or “the natural logarithm of x .”

The logarithm y is the exponent to which e must be raised to get x .

Since the functions $y = e^x$ and $y = \ln(x)$ are inverse functions, $\ln(e^x) = x$ for all x and $e^{\ln(x)} = x$ for $x > 0$.

How To...

Given a natural logarithm with the form $y = \ln(x)$, evaluate it using a calculator.

1. Press **[LN]**.
2. Enter the value given for x , followed by **[)]**.
3. Press **[ENTER]**.

Example 8 **Evaluating a Natural Logarithm Using a Calculator**

Evaluate $y = \ln(500)$ to four decimal places using a calculator.

Solution

- Press **[LN]**.
- Enter 500, followed by **[)]**.
- Press **[ENTER]**.

Rounding to four decimal places, $\ln(500) \approx 6.2146$

Graphs of Logarithms

In **Graphs of Exponential Functions**, we saw how creating a graphical representation of an exponential model gives us another layer of insight for predicting future events. How do logarithmic graphs give us insight into situations? Because every logarithmic function is the inverse function of an exponential function, we can think of every output on a logarithmic graph as the input for the corresponding inverse exponential equation. In other words, logarithms give the *cause* for an *effect*.

To illustrate, suppose we invest \$2,500 in an account that offers an annual interest rate of 5%, compounded continuously.

We already know that the balance in our account for any year t can be found with the equation $A = 2500e^{0.05t}$.

But what if we wanted to know the year for any balance? We would need to create a corresponding new function by interchanging the input and the output; thus we would need to create a logarithmic model for this situation. By graphing the model, we can see the output (year) for any input (account balance). For instance, what if we wanted to know how many years it would take for our initial investment to double? **Figure 3** shows this point on the logarithmic graph.

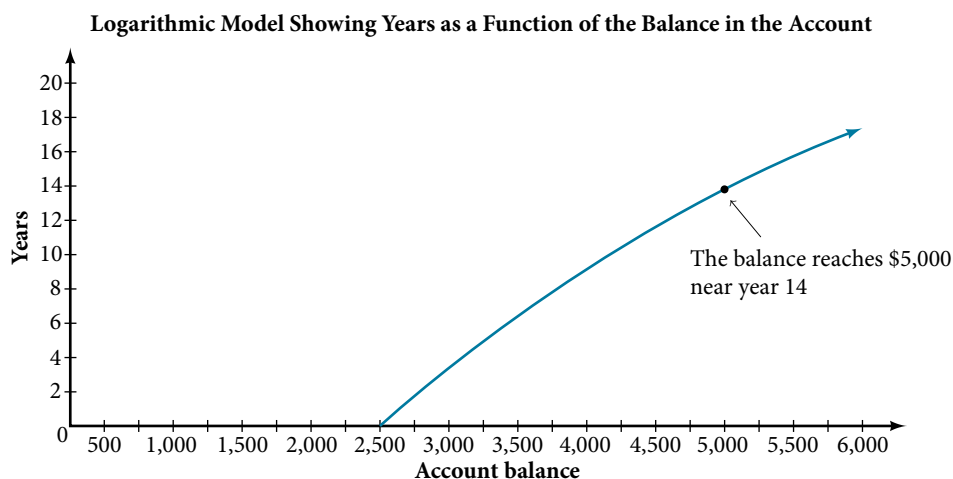


Figure 3

Finding the Domain of a Logarithmic Function

Before working with graphs, we will take a look at the domain (the set of input values) for which the logarithmic function is defined.

Recall that the exponential function is defined as $y = b^x$ for any real number x and constant $b > 0$, $b \neq 1$, where

- The domain of y is $(-\infty, \infty)$.
- The range of y is $(0, \infty)$.

In the last section we learned that the logarithmic function $y = \log_b(x)$ is the inverse of the exponential function $y = b^x$. So, as inverse functions:

- The domain of $y = \log_b(x)$ is the range of $y = b^x$: $(0, \infty)$.
- The range of $y = \log_b(x)$ is the domain of $y = b^x$: $(-\infty, \infty)$.

How To...

Given a logarithmic function, identify the domain.

1. Set up an inequality showing the argument greater than zero.
 2. Solve for x .
 3. Write the domain in interval notation.
-

Example 9 Identifying the Domain of a Logarithm

What is the domain of $f(x) = \log_2(x + 3)$?

Solution The logarithmic function is defined only when the input is positive, so this function is defined when $x + 3 > 0$.

Solving this inequality,

$$x + 3 > 0 \quad \text{The input must be positive.}$$

$$x > -3 \quad \text{Subtract 3.}$$

The domain of $f(x) = \log_2(x + 3)$ is $(-3, \infty)$.

Example 10 Identifying the Domain of a Logarithm

What is the domain of $f(x) = \log(5 - 2x)$?

Solution The logarithmic function is defined only when the input is positive, so this function is defined when $5 - 2x > 0$.

Solving this inequality,

$$5 - 2x > 0 \quad \text{The input must be positive.}$$

$$-2x > -5 \quad \text{Subtract 5.}$$

$$x < \frac{5}{2} \quad \text{Divide by } -2 \text{ and switch the inequality.}$$

The domain of $f(x) = \log(5 - 2x)$ is $(-\infty, \frac{5}{2})$.

Graphing Logarithmic Functions

Now that we have a feel for the set of values for which a logarithmic function is defined, we move on to graphing logarithmic functions. The family of logarithmic functions includes the parent function $y = \log_b(x)$ along with all its transformations: shifts, stretches, compressions, and reflections.

We begin with the parent function $y = \log_b(x)$. Because every logarithmic function of this form is the inverse of an exponential function with the form $y = b^x$, their graphs will be reflections of each other across the line $y = x$. To illustrate this, we can observe the relationship between the input and output values of $y = 2^x$ and its equivalent $x = \log_2(y)$ in **Table 1**.

x	-3	-2	-1	0	1	2	3
$2^x = y$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$\log_2(y) = x$	-3	-2	-1	0	1	2	3

Table 1

Using the inputs and outputs from **Table 1**, we can build another table to observe the relationship between points on the graphs of the inverse functions $f(x) = 2^x$ and $g(x) = \log_2(x)$. See **Table 2**.

$f(x) = 2^x$	$(-3, \frac{1}{8})$	$(-2, \frac{1}{4})$	$(-1, \frac{1}{2})$	(0, 1)	(1, 2)	(2, 4)	(3, 8)
$g(x) = \log_2(x)$	$(\frac{1}{8}, -3)$	$(\frac{1}{4}, -2)$	$(\frac{1}{2}, -1)$	(1, 0)	(2, 1)	(4, 2)	(8, 3)

Table 2

As we'd expect, the x - and y -coordinates are reversed for the inverse functions. **Figure 4** shows the graph of f and g .

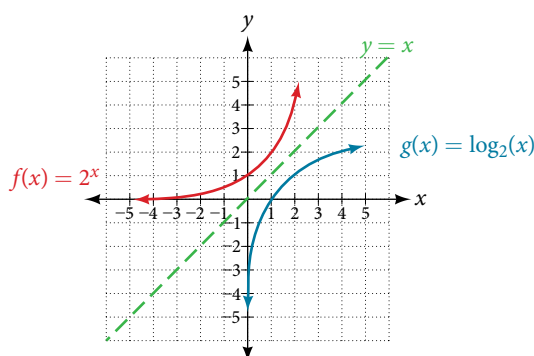


Figure 4 Notice that the graphs of $f(x) = 2^x$ and $g(x) = \log_2(x)$ are reflections about the line $y = x$.

Observe the following from the graph:

- $f(x) = 2^x$ has a y -intercept at $(0, 1)$ and $g(x) = \log_2(x)$ has an x -intercept at $(1, 0)$.
- The domain of $f(x) = 2^x$, $(-\infty, \infty)$, is the same as the range of $g(x) = \log_2(x)$.
- The range of $f(x) = 2^x$, $(0, \infty)$, is the same as the domain of $g(x) = \log_2(x)$.

characteristics of the graph of the parent function, $f(x) = \log_b(x)$

For any real number x and constant $b > 0$, $b \neq 1$, we can see the following characteristics in the graph of $f(x) = \log_b(x)$:

- one-to-one function
- vertical asymptote: $x = 0$
- domain: $(0, \infty)$
- range: $(-\infty, \infty)$
- x -intercept: $(1, 0)$
and key point $(b, 1)$
- y -intercept: none
- increasing if $b > 1$
- decreasing if $0 < b < 1$

See **Figure 5**.

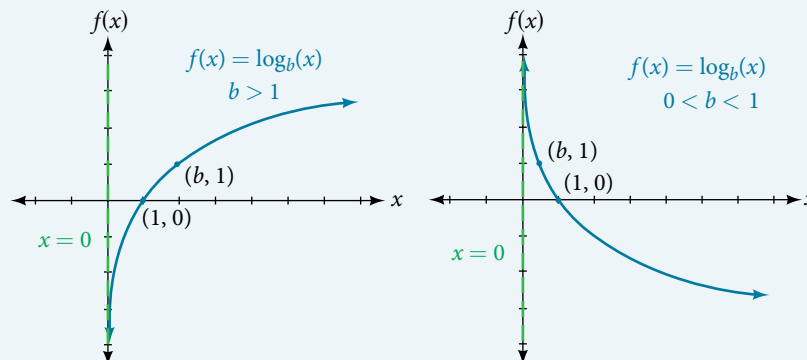


Figure 5

How To...

Given a logarithmic function with the form $f(x) = \log_b(x)$, graph the function.

1. Draw and label the vertical asymptote, $x = 0$.
2. Plot the x -intercept, $(1, 0)$.
3. Plot the key point $(b, 1)$.
4. Draw a smooth curve through the points.
5. State the domain, $(0, \infty)$, the range, $(-\infty, \infty)$, and the vertical asymptote, $x = 0$.

Example 11 Graphing a Logarithmic Function with the Form $f(x) = \log_b(x)$.

Graph $f(x) = \log_5(x)$. State the domain, range, and asymptote.

Solution Before graphing, identify the behavior and key points for the graph.

- Since $b = 5$ is greater than one, we know the function is increasing. The left tail of the graph will approach the vertical asymptote $x = 0$, and the right tail will increase slowly without bound.
- The x -intercept is $(1, 0)$.
- The key point $(5, 1)$ is on the graph.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points (see **Figure 6**).

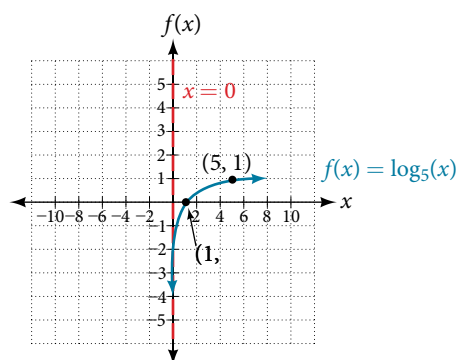


Figure 6

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.